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Regular and chaotic motion in some quartic potentials

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Abstract. The Hamiltonian $H_\alpha(x, y, p_x, p_y) = \frac{1}{2}(p_x^2 + p_y^2) + V_\alpha(x, y)$ with $0 \leq \alpha \leq 1$ and

$$V_\alpha(x, y) = \tfrac{1}{24}[(x+y)^4 + (x-y)^4] - \tfrac{1}{24}\alpha(x^4 + y^4)$$

is integrable for $\alpha = 0$, and for $\alpha = 1$ the potential is

$$V_1(x, y) = \tfrac{1}{2}x^2y^2.$$

A detailed numerical study using surfaces of section and properties of periodic orbits for the entire range of α strongly indicates that there are no invariants for H_1, so the motion is completely chaotic. This result presents problems for the semiclassical quantisation of gauge fields.

1. Introduction

A conservative dynamical system with Hamiltonian of the form

$$H(\boldsymbol{r}, \boldsymbol{p}) = \tfrac{1}{2}\boldsymbol{p} \cdot \boldsymbol{p} + V(\boldsymbol{r}), \qquad (\boldsymbol{r} = x_1, \ldots, x_{11}; \boldsymbol{p} = p_1, \ldots, p_n), \tag{1.1}$$

where

$$V(\lambda \boldsymbol{r}) = \lambda^k V(\boldsymbol{r}), \qquad k \neq 0, \tag{1.2}$$

is a homogeneous potential of degree k, exhibits mechanical similarity (Landau and Lifshitz 1969). Corresponding to any motion of energy $E \neq 0$, there is a similar motion for all other energies of the same sign. The properties of the motion for any positive energy E can be determined by simple scaling from the properties of the motion on the energy shell $E = 1$.

For convenience we sometimes describe the system as a particle of unit mass confined to the x–y plane and moving in the potential $V(x, y)$, where x and y are rectangular configuration coordinates.

Many important dynamical systems can be brought into this form. The simplest of these that can exhibit irregular or chaotic motion are the systems of two degrees of freedom with quartic potentials. If we suppose in addition that the systems are symmetric for reflections about two perpendicular axes, then we can choose configuration coordinates so that the Hamiltonian has the form

$$H_\alpha(x, y, p_x, p_y) = \tfrac{1}{2}(p_x^2 + p_y^2) + V_\alpha(x, y)$$

where

$$V_\alpha(x, y) = \tfrac{1}{24}[(x+y)^4 + (x-y)^4] - \tfrac{1}{12}\alpha(x^4 + y^4) \tag{1.3}$$

and $0 \leq \alpha \leq 1$.

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If the (x, y) axes are rotated by 45° to become (ξ, η) then (1.3) transforms to

$$H'_\alpha(\xi, \eta, p_\xi, p_\eta) = \tfrac{1}{2}(p_\xi^2 + p_\eta^2) + V'_\alpha(\xi, \eta)$$

where

$$V'_\alpha(\xi, \eta) = \tfrac{1}{6}(\xi^4 + \eta^4) - \tfrac{1}{48}\alpha[(\xi - \eta)^4 + (\xi + \eta)^4], \tag{1.4}$$

and $0 \leqslant \alpha \leqslant 1$, so that at $\alpha = 0$ the system is separable into two independent quartic oscillators. The motion is completely regular and always confined to invariant tori.

For $\alpha = 1$ the Hamiltonian has the form

$$H_1(x, y, p_x, p_y) = \tfrac{1}{2}(p_x^2 + p_y^2 + x^2 y^2). \tag{1.5}$$

This is a particularly interesting system, and the principal object of our study. Its elementary properties are summarised in § 2, and in addition to its intrinsic interest, it has been suggested as a very simplified zero-dimensional model of the classical Yang–Mills field.

For this reason it has been studied numerically and no evidence of regular motion has been found (Martinyan *et al* 1981).

We have carried out a systematic search for signs of regular motion in this system, by studying the one-parameter set of Hamiltonians (1.3) and equivalently (1.4) and then following the fate of the regular regions of phase space as α tends to 1. Our investigations strongly suggest that there is no regular region and that the motion is always irregular except for special orbits, like the periodic orbits that form a set of measure zero.

No satisfactory and practical means is known of using semiclassical mechanics to determine the energy levels of large irregular or chaotic regions of phase space. This suggests, though of course it does not prove, that the same might also be true of the Yang–Mills field.

2. Motion in a $x^2y^2/2$ potential

The contours of this potential are illustrated in figure 1(*d*). At any positive energy there are four channels down which a particle can pass in the neighbourhoods of the positive and negative x and y axes. If the particle moves along either axis, it escapes to infinity.

But this motion is exceptional. If the particle goes down a channel along any other orbit, we can determine what happens approximately by using the principle of adiabatic invariance. Suppose for example that the particle moves into the positive x channel, and consider the motion in the y direction. It can be represented by the Hamiltonian

$$H_y(y, p_y) = \tfrac{1}{2}(p_y^2 + x^2 y^2). \tag{2.1}$$

If x is considered as a parameter then this represents a linear oscillator of angular frequency $\omega_y = x$.

Far down the channel, the relative change in x over one period of this motion is small, so the action

$$I_y = E_y/\omega_y = E_y/x \tag{2.2}$$

is an adiabatic invariant. The effective Hamiltonian for the mean motion along the

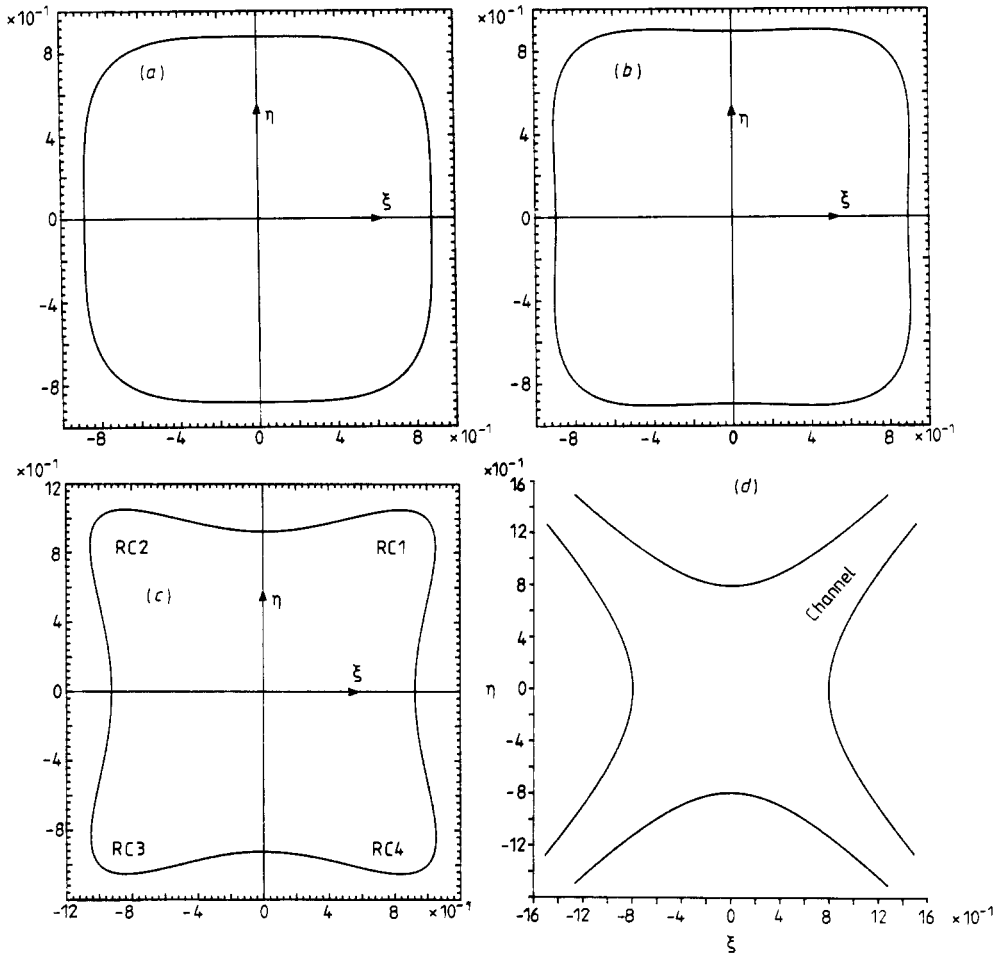

Figure 1. Depicts the evolution with increasing α of the potential contour in the Hamiltonian system of (1.4). (a), (b), (c), (d) respectively represent the potential at $\alpha = 0, 0.25, 0.7, 1.0$.

channel is then

$$H_x^{\text{eff}}(x, p_x) \simeq \tfrac{1}{2}p_x^2 + I_y x. \tag{2.3}$$

As the particle moves into the channel, it loses energy from the x motion into the y motion, and is reflected back towards the origin when

$$x = E/I_y \tag{2.4}$$

where E is the total energy. The mechanism is similar to that of a magnetic bottle for charged particles.

However, the volume of phase space down any channel diverges logarithmically, so the statistics of the motion is unusual. There can be no ergodic behaviour in the usual sense, because although the particle is eventually reflected, an attempt to explore the whole of phase space available to it results on average in a higher and higher proportion spent in the channels by comparison with the region around the origin as

the motion continues, despite the fact that it continually returns to this central region. In practice this behaviour causes difficulties for the numerical integrations, for long periods spent in the channels teach us little about the overall nature of the motion.

In the region around the origin, the potential has four convex reflecting hills between the channels and there are various types of symmetric periodic orbits between them. For example, there is motion in a straight line through the origin between opposite hills, for which the motion is obviously unstable. Furthermore, preliminary numerical integrations by Martinyan *et al* (1981) have failed to find any stable periodic orbits or invariant tori for this system. Their apparent absence suggests that there may be no regular motion. However, the regions which can contain regular motion can be very small (Contopolous 1970), and such regions have been found by studying the dependence of the regular orbits upon a parameter. Usually this parameter has been the energy, but in the case of the $x^2y^2/2$ potential the energy parameter changes only the scale of the motion, it does not affect its character. So in this case we vary the parameter α of the equations (1.3) and (1.4), for fixed energy E.

When $\alpha = 0$ the system is integrable and the whole of the phase space is regular. As α increases, the regular region breaks up into smaller regular regions and irregular regions. We continue following the regular regions with increasing α to determine if they exist in the $x^2y^2/2$ potential, where $\alpha = 1$.

3. Techniques and formalism

Here we describe two methods to distinguish regular and irregular motion. Both of these involve considering the surface of section, defined by keeping one configuration coordinate equal to zero and its canonically conjugate momentum greater than zero. With the (1.3) and (1.4) forms of the parametrisation, the surfaces of section we take are respectively

$$x = 0, \qquad p_x > 0, \tag{3.1a}$$

$$\xi = 0, \qquad p_\xi > 0. \tag{3.1b}$$

In addition we consider the area preserving maps T_x, T_ξ, defined on (3.1a), (3.1b) respectively, and generated by the Hamiltonian phase flow from these surfaces to themselves. As the discussion in the rest of this section applies equally to (3.1a) and (3.1b) we consider the general surface of section S and the corresponding area preserving map T.

The first method of investigation, originally suggested by Poincaré, and explained in detail by Berry (1978), Hénon and Heiles (1963), Helleman (1980) and others, is to consider the repeated action of T upon an initial point in the surface of section. If after a sufficient number of iterates, the resulting points form a closed curve, called an invariant curve, the trajectory corresponding to them lies on an invariant torus or KAM surface. If, instead, these points are dense in a two-dimensional area in the plane then the trajectory corresponding to them is irregular.

This method provides an overall picture of the motion, but is unsuitable for investigating small regular regions of phase space. For such investigations, it is better to study the stability of periodic orbits following Greene (1979).

Suppose that the periodic orbit $Z_0(t)$ has K distinct intersections with the surface of section S, one of which we call O. Then we first calculate the 2×2 matrix M, which is the linearisation of T^K about O.

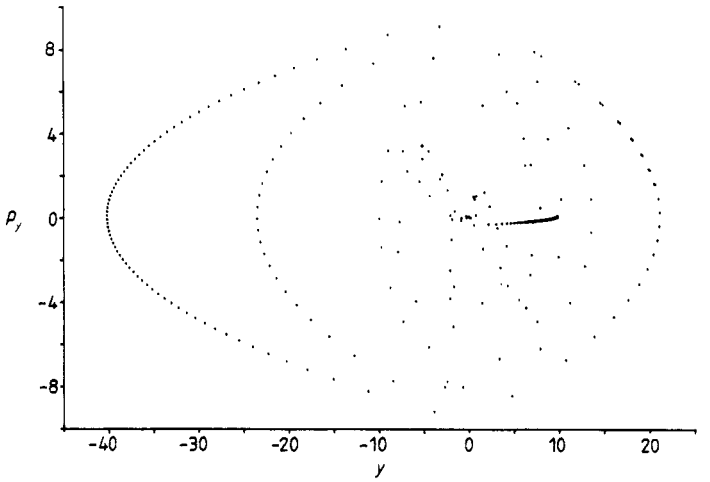

Figure 2. 300 intersections of one trajectory at $\alpha = 1$ with the surface of section of (3.1*a*).

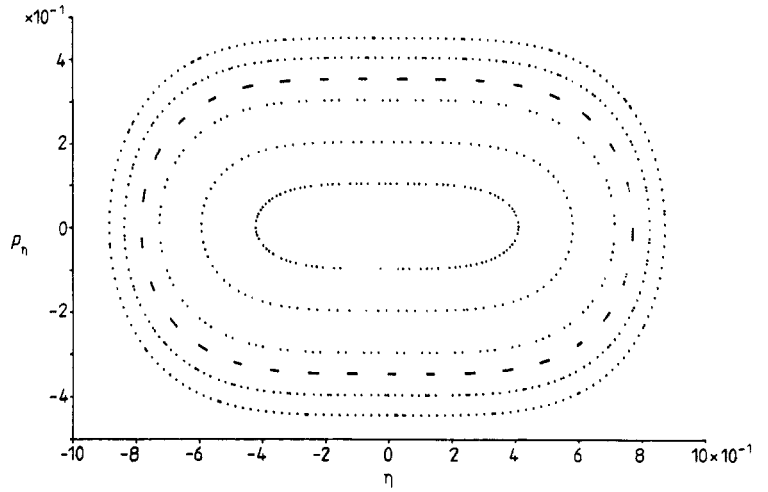

Figure 3. The surface of section of (3.1*b*) at $\alpha = 0$.

The residue, R, is defined in terms of the matrix M by the relation

$$R = \tfrac{1}{4}(2 - \mathrm{Tr}\, M) \tag{3.2}$$

where Tr M is the trace of M.

If $0 < R < 1$, there are closed invariant curves immediately surrounding the fixed points corresponding to $Z_0(t)$ in the surface of section. All these points are elliptic fixed points and $Z_0(t)$ is stable, as trajectories initially close to it remain bound to it for all time.

If, however, $R < 0$ or $R > 1$, then these points are hyperbolic fixed points and the trajectory is unstable. The cases of equality, $R = 0, 1$, are primarily of interest as defining a boundary between the other cases.

We have established two methods of investigating the structure of phase space; the first one is suitable for investigating the macroscopic structure, and the second for investigating the microscopic structure. In later sections we describe how these two methods are applied to the quartic potentials of the introduction.

4. Classification of periodic orbits

Periodic orbits must be found and classified before they can be used successfully to determine the structure of phase space. The choice of the ones to be studied in this paper arose out of the properties of symmetry of the potential associated with the parametrised system (1.4).

We consider three main types of periodic orbit A, B, C, in detail and note the existence of a fourth class D. Orbits of type A have two distinct points along them at which $p_\xi = p_\eta = 0$. Orbits of types B and C are symmetrical about the diagonal and ξ axes of figure (1*a*), and type D periodic orbits are invariant when reflected first about one coordinate axis and then the other. The sets A, B, C and D are not necessarily disjoint, for example, at $\alpha = 0$ A is a subset of the union of the sets C and D.

Periodic orbits are further categorised by their resonance numbers N_ξ and N_η, which for an individual orbit are respectively the number of oscillations in the ξ and η directions during one period. For the rest of the paper such orbits will be referred to as $N_\xi : N_\eta$ periodic orbits.

We further subdivide the classes A, B, C into sets dependent upon the N_ξ resonance numbers of the periodic orbits in them. For example the set A_N consists of type A periodic orbits with N_ξ equal to N. The sets B_N and C_N of types B and C periodic orbits respectively are defined in the same fashion.

To locate periodic orbits belonging to A_N we consider T_1, the map generated by the Hamiltonian phase flow of the points in the surface $\{\boldsymbol{x} : p_\xi = 0\}$ back to itself. Then the function $P^{(N)}(\xi^{(0)})$ is defined as the η component of the momentum of the image of a point under T_1^N, with original coordinates $(\xi^{(0)}, \eta^{(0)}, 0, 0)$. Any root of $P^{(N)}(\xi^{(0)})$ corresponds to the initial condition of a type A periodic orbit with N as its N_ξ resonance number.

With orbits in C_N, the function $P^{(N)}(\xi^{(0)})$ is defined to be the ξ component of the momentum of the image of a point under T_2^N, with original coordinates $(\xi^{(0)}, 0, 0, p_\eta^{(0)})$. T_2 is the map generated by the Hamiltonian phase-flow from the surface $\{\boldsymbol{x} : \eta = 0\}$ back to itself.

After rotating the configuration coordinate axes so that they become the diagonal axes of diagram 1(*a*), the function $P^{(N)}(x^{(0)})$ is defined in the same way for orbits in B_N as $P^{(N)}(\xi^{(0)})$ was defined for orbits in C_N. In both cases the roots of $P^{(N)}$ are the initial conditions of periodic orbits with N as their first resonance numbers. With orbits of types A, B and C the function is continuous and differentiable.

Typically the residue of a fixed point starts at zero when the point is created and increases until the point bifurcates to produce another of twice the period when the residue passes through 1 (Greene *et al* 1981). However, because of the special symmetries of the orbits in the classes A, B, C, D the residue of a fixed point behaved differently: it increased from zero to one and then decreased, the periodic point bifurcating to produce two more stable periodic points of the same period when the residue passed through zero. These symmetries can often speed the calculation of the residue. Further details are given by Greene (1979), DeVogelaere (1954) and Green *et al* (1981).

5. Surface of section method

We first apply the surface of section method to the $x^2y^2/2$ potential, showing the macroscopic phase space structure and the necessity of using alternative procedures.

The calculations were performed using either a Runge–Kutta or Adams technique. The resulting numerical errors were checked by comparing runs made with the same initial conditions at different step-lengths.

At $\alpha = 1$ we used both the (3.1a) and (3.1b) surface of sections, which at this value of α are respectively unbounded and bounded.

As described in § 4, we used symmetry to find various classes of periodic orbits in the $x^2y^2/2$ potential, and then looked for regular regions associated with stable periodic orbits.

Twenty sets of initial conditions apparently generated non-periodic irregular trajectories, each one being qualitatively similar to any other over a sufficient period of time. Typically the action of (2.2) was approximately conserved over most of the length of an integrated trajectory, showing that much of its motion was down one channel or another. Figure 2 exemplifies this. It is the plot of 300 intersections of a

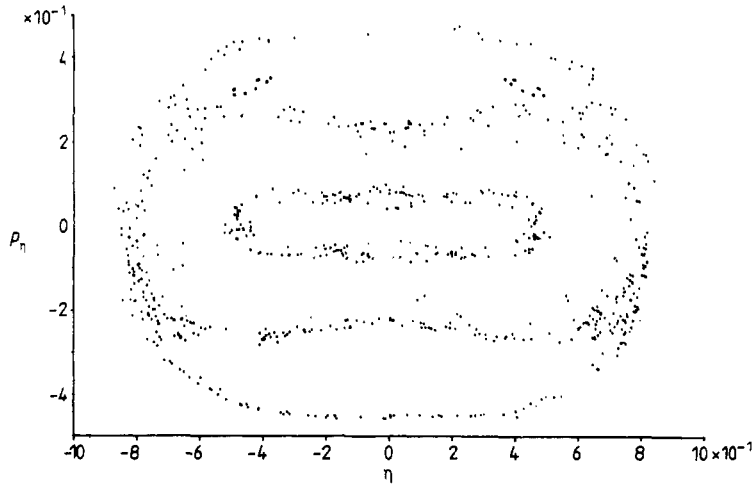

Figure 4. A macroscopic examination of (3.1b) at $\alpha = 0.25$, showing both regular and irregular regions.

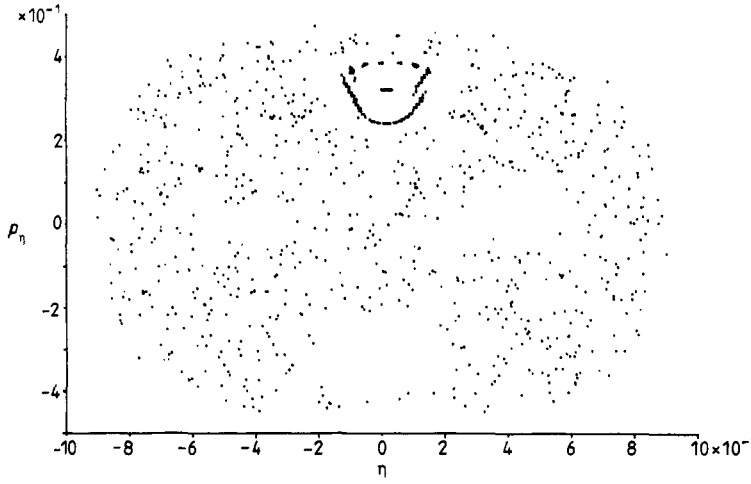

Figure 5. A macroscopic examination of (3.1b) at $\alpha = 0.5$, showing smaller regular regions than at $\alpha = 0.25$.

trajectory with the surface of section given by (3.1*a*). When the motion is down a channel along the y axis, consecutive intersections of the trajectory with the surface of section form an open curve of points. All plots of intersections of trajectories with the surface of section of (3.1*b*) showed no such curves, thus indicating that the trajectories are irregular.

Stable periodic orbits are detected at values of $\alpha < 1$ by investigating regular regions existing at these values in the system (1.3) and equivalently (1.4). In addition we followed such regions with increasing α to determine the extent of regular motion in the $x^2y^2/2$ potential. In this work we consider only the (1.4) form of the parametrisation with the surface of section (3.1*b*).

Consider first the surface of section at $\alpha = 0$. The system is integrable with all motions lying on invariant tori. Figure 3 exemplifies this, showing only concentric invariant curves representing rational and irrational tori. At $\alpha = 0.25$, many of the initial conditions which generate invariant curves at $\alpha = 0.0$ now generate irregular ones. A macroscopic view of the surface of section, figure 4, shows both large irregular regions and also many of low stochasticity. However, by $\alpha = 0.5$, most of these with the exception of two have dwindled, so that on the macroscopic scale of figure 5 they are hardly recognisable without prior knowledge of their existence. At $\alpha = 0.8$, most of the initial conditions which generated invariant curves at $\alpha = 0.25$ now generate irregular ones. Figure 6 shows that there are still two obvious regular regions. These are associated with the cusps, RC_i, $i = 1, 2, 3, 4$, of the potential contour in figure 1(*c*). But by $\alpha = 0.93$, figure 7 shows that these too have shrunk and there do not appear to be other regular regions left. Finally, at $\alpha = 0.99$, we did not observe any regular regions.

The above results show the difficulty of detecting regular motion in the $x^2y^2/2$ potential, applying the surface of section method. Despite the unsuitability of the method to finding small regular regions, it is still useful for investigating the large scale structure of phase space for $\alpha < 1$. This proved useful in detecting the periodic orbits considered in the next section.

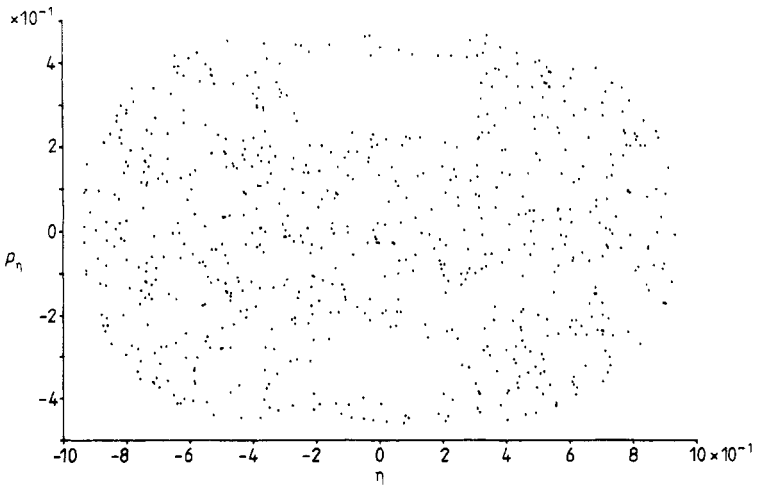

Figure 6. A macroscopic examination of (3.1*b*) at $\alpha = 0.8$. There are two large regular regions centred around $\eta = 0$, $P\eta = \pm 0.3162$.

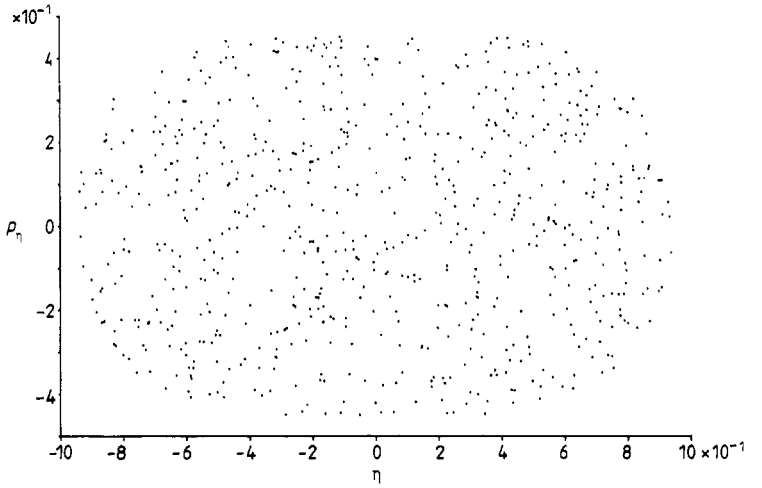

Figure 7. A macroscopic examination of (3.1*b*) at $\alpha = 0.93$, showing that no large regular regions exist at this parameter value.

6. Results for periodic orbits

As described in § 1, we followed stable periodic orbits of types A, B and C in the system (1.4) created at values of $\alpha < 1$, to seek regular trajectories in this system at $\alpha = 1$ when (1.4) takes the form

$$H = \tfrac{1}{2}(p_\xi^2 + p_\eta^2) + \tfrac{1}{8}(\xi^2 - \eta^2)^2. \tag{6.1}$$

Initially we considered type A periodic orbits created at $\alpha = 0$. The residues of eight short periodic orbits from each of the sets A_i, $i = 1, 2, 3$, all increased from zero up till 1 and then down past zero as α was increased from zero with

$$\alpha_{JK} < \alpha_{JL}, \qquad J = 1, 2, 3, \qquad K > L \geqslant J+1, \tag{6.2}$$

where K, L are chosen from the eight smallest integers coprime with J. α_{JL} is the value of α at which the $J:L$ type periodic orbit becomes unstable, that is when the value of its residue becomes less than zero.

Associated with each of the $J:L$ orbits, $L \neq J+1$, is a value of α, α_{cr}, at which we noted a large increase in numerical errors from the integration procedures, whatever the interval of integration. In addition, by considering the type A periodic orbits with resonance numbers $J:J+1$, $1 < J < 9$, we found that

$$\alpha_{LL+1} < \alpha_{KK+1}, \qquad 1 \leqslant K \leqslant L \leqslant 9. \tag{6.3}$$

Regarding table 1 we see that $\alpha_{12} = 0.38$. Assuming that (6.2), (6.3) hold for all type A periodic orbits created at $\alpha = 0$, we suggest that they all bifurcate at values of $\alpha < 0.4$. The quantity $a_{NM}(0)$ in table 1 is the root of $P^{(N)}$ corresponding to the $N:M$ type A periodic orbit at $\alpha = 0$.

Type A periodic orbits with resonance numbers $N:2M$ are also of type C and these bifurcate, usually to produce two more type C orbits of the same period, which are not of type A. We consider these bifurcations later when we examine periodic orbits created at values of $\alpha > 0$. Those type A periodic orbits created with resonance

Table 1. Values of α, α_{NM}, at which $N:M$, $N+1=M$, type A periodic orbits created at $\alpha=0$ become unstable.

$N:M$	α_{NM}	$a_{NM}(0)$
1:2	0.38	0.433
3:4	0.245	0.6163
4:5	0.215	0.6462
5:6	0.195	0.6647
6:7	0.175	0.6772
7:8	0.155	0.6862
8:9	0.14	0.6929
9:10	0.125	0.6983

numbers $N:(2M+1)$, N odd and coprime with $2M+1$, bifurcate to produce two periodic orbits of type D which we did not examine in detail, but there are indications that these bifurcate before $\alpha=1$.

We consider next stable periodic orbits of types A, B, C created at values of $\alpha>0$. Such orbits are located by the momentum function $P^{(N)}$; this has a low gradient and a smooth simple structure about roots corresponding to stable periodic orbits, whilst most roots in a region with large densities of roots, the gradients of the function about these being usually steep, correspond to unstable periodic orbits. For example figure 8 shows that the function $P^{(4)}$, defined for class A_4 orbits at $\alpha=0.5$, possesses regions of large root density such as that marked DR, where stable periodic orbits are few and difficult to locate, whilst the region marked CR possesses a single root corresponding to a stable periodic orbit.

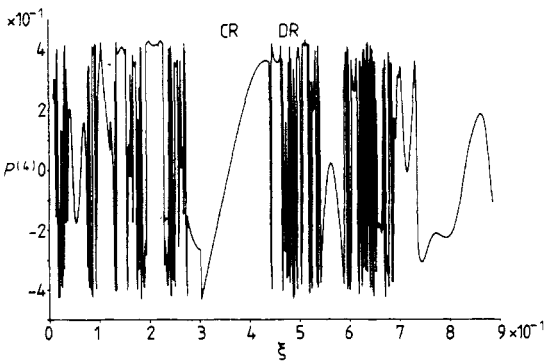

Figure 8. $P^{(4)}$ defined for A_4 at $\alpha=0.5$.

Moreover, stable type B periodic orbits are often found in the largest regular regions associated with the system (1.4), at values of α close to 1; hence we followed such orbits to determine the fates of these regions after they became too small to locate using the surface of section method.

We define a parameter R_s, where for an individual periodic orbit

$$R_s=(\alpha_1-\alpha_0)/(1-\alpha_0) \tag{6.4}$$

where α_0 is the value of the parameter at which the periodic orbit is created and α_1

is the value of the parameter at the first bifurcation. If $0 < R_s < 1$ the periodic orbit bifurcates at a value of $\alpha < 1$, if $1 < R_s$ the periodic orbit is stable at $\alpha = 1$.

We concentrate on the sets A_2, B_4 consisting of periodic orbits which are shorter than most of those belonging to the higher-order sets A_i, B_j, $i > 2$, $j > 4$ respectively. Typically the shorter the length of a periodic orbit the larger its interval of stability over α will be. For example in table 3, which shows quantities for orbits in B_4, rows marked with an asterisk correspond to periodic orbits also in B_2 and B_1; these have larger values of R_s and are shorter than the other orbits considered. So it is possible to discover the stability characteristics of many periodic orbits of types A and B from those of orbits in A_2 and B_4 respectively.

None of the periodic orbits belonging to A_2 and B_4 considered in tables 2 and 3 respectively are stable at $\alpha = 1$ and thus none of the listed values of R_s are larger than 1, moreover there are no trends in either sets of R_s which could imply that orbits in A_2 or B_4 created at values of α larger than the ones considered will be stable at $\alpha = 1$. Hence as orbits of types A_2 and B_4 are shorter than most other periodic orbits of types A and B respectively, we suggest that most periodic orbits of these types created at values of $\alpha < 1$ bifurcate before $\alpha = 1$.

Greene *et al* (1981) have found that typically periodic orbits in reversible area preserving maps lose their stability when the map is perturbed and bifurcate to produce a periodic orbit of twice the period or two of the same period. An infinite sequence of bifurcations is accomplished, accumulating at a finite value of the parameter, sometimes referred to as the accumulation point of the original orbit.

The observed bifurcations of type B periodic orbits produced more type B periodic orbits, so the above analysis suggests that the accumulation points of these orbits are

Table 2. Values of α, α_0 and α_1, at which periodic orbits in A_2 are created and destroyed respectively. We show also the corresponding values of R_s and $a_{2M}(\alpha_0)$. This last quantity is the root of $P^{(2)}$ corresponding to the $2:M$ periodic orbit at $\alpha = \alpha_0$.

α_0	α_1	$a_{2M}(\alpha_0)$	M	R_s
0.004	0.56	0.03	1	0.56
0.818 314	0.8222	0.170 07	2	0.022
0.833 6	0.8545	0.218 55	2	0.126
0.928 1	0.9372	0.846 41	1	0.127

Table 3. Values of α, α_0 and α_1 at which periodic orbits in B_4 are created and destroyed respectively; also the corresponding values of R_s and $b_{4M}(\alpha_0)$. The last quantity is the root of $P^{(4)}$ corresponding to the $4:M$ periodic orbit at $\alpha = \alpha_0$. Rows marked with an asterisk correspond to periodic orbits also in B_2 or B_1.

α_0	α_1	$b_{4M}(\alpha_0)$	M	R_s	
0.92	0.924 16	0.031	34	0.52	*
0.925 7	0.930 4	0.0044	18	0.07	*
0.930 4	0.932 2	0.158 71	18	0.04	*
0.979 6	0.979 958	0.0873	68	0.01	
0.983 94	0.984 21	0.004 15	156	0.017	
0.986 9	0.987 12	0.023	86	0.017	
0.989	0.989 213	0.013 4	94	0.019	

at values of $\alpha < 1$. However, type A periodic orbits which are of types C or D also, bifurcate to produce respectively stable periodic orbits of types C or D, but which are not of type A. Thus to gain information about the bifurcations of some type A periodic orbits, we examined stable type C periodic orbits created at values of $\alpha > 0$. We were unable to locate sufficient numbers of these to construct a table of quantities, as we were able to do with periodic orbits of types B_4 and A_2. However, the values of R_s of the type C orbits we did investigate were all less than 0.14.

A conclusion about the fates of most type C periodic orbits cannot be reached in the same way as it was for orbits of types A and B, but the rareness of stable type C periodic orbits and the finding that the observed values of R_s are less than 1 does imply that only a few, if any, type C periodic orbits created at values of $\alpha < 1$ are stable when $\alpha = 1$.

On the basis of the above, we suggest that the accumulation point of any sequence of bifurcations which consist of orbits only belonging to the sets A, B or C is at $\alpha < 1$. Greene *et al* (1981) showed that the ratios between successive bifurcations in any sequence tended to 8.7. Assuming that such ratios in most of the sequences involved in this paper are approximately this value, then the accumulation point of a sequence produced by a particular periodic orbit is at $\alpha < 1$ if the value of R_s for this orbit is less than 0.14. This we observed always to be true for orbits of types A, B or C created at values of $\alpha > 0.5$. This suggests that even if some of the orbits are not of types A, B or C in a sequence of bifurcations of a periodic orbit, itself one of these types, the sequence still has an accumulation point at $\alpha \leqslant 1$.

By studying stable periodic orbits we have detected much smaller regular regions than we were able to using the surface of section technique. Nevertheless all the indications are that regular motion disappears before $\alpha = 1$.

7. Discussion and conclusions

We have carried out an exhaustive numerical analysis of the orbits of the systems (1.3), using two separate surfaces of section to observe the broad structure of the regular and irregular regions and three classes of periodic orbits to follow the fine structure of the regular regions as $\alpha \to 1$. Every calculation is consistent with the conclusions that the motion of a particle in the potential $\frac{1}{2}x^2y^2$ is irregular everywhere except for a set of measure zero.

This potential may be considered as a very reduced model of a Yang–Mills field. Since there is no known practical method for the semiclassical quantisation of irregular motion, the same might be true of such fields.

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